AD645545

# ASYMPTOTICALLY OPTIMAL STATISTICS IN SOME MODELS WITH INCREASING FAILURE RATE AVERAGES
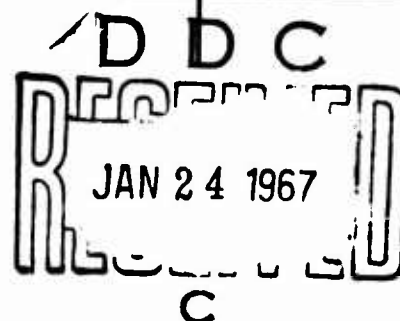
by

Kjell Doksum

# OPERATIONS RESEARCH CENTER

COLLEGE OF ENGINEERING

D D C

JAN 2 4 1967

C

# UNIVERSITY OF CALIFORNIA-BERKELEY

ASYMPTOTICALLY OPTIMAL STATISTICS IN SOME MODELS
WITH INCREASING FAILURE RATE AVERAGES


by

Kjell Doksum
Operations Research Center
University of California, Berkeley

# ABSTRACT

Let  F  and  G  be defined by  $F(t) = H(\gamma t)$  and  $G(t) = H(\theta t)$  where  H  is unknown and  $H(0) = 0$ .
For testing the equality of the means of  F
and  G  in the two-sample problem; it is shown
that the Savage (The Annals of Mathematical Statistics
(1956) pp 590-615) statistic maximizes the
minimum power over  IFRA(or IFR)  distributions
asymptotically.  Asymptotic uniqueness holds
only in a class of rank tests.  The results are
extended to censored samples, the problem of
estimating the ratio of the means, and the
k-sample problem.

1.  <u>INTRODUCTION AND SUMMARY.</u>  Birnbaum, Esary and Marshall (1966) have
shown that the class $\mathcal{F}$ of distributions with increasing failure rate averages
(IFRA)  characterizes the concept of wear-out in the sense that $\mathcal{F}$ is the
smallest class that contains the exponential distributions and is closed
under the formation of coherent systems.

In this note, statistical inference for models in which the distributions
are unknown and  IFRA  will be considered.  Let  F  and  G  be defined by

$$(1.1) \qquad\qquad F(t) = H(t/\theta) \quad \text{and} \quad G(t) = H(t/\gamma)$$

where  H  is an unknown  IFRA  distribution with  $H(0) = 0$ .  Then, for the
two-sample problem where one tests the equality of the means of  F  and  G ,
it is shown that the Savage (1956) statistic maximizes the minimum power over
IFRA  distributions asymptotically.  This asymptotic minimax solution is
extended to censored samples and it turns out that the Gastwirth (1965) modified
version of the Savage statistic is asymptotically minimax for this case.
Asymptotic uniqueness of these minimax solutions holds only in a class of
rank tests.  The results are extended to obtain an estimate of the ratio of
the means that minimizes the maximum asymptotic variance over  IFRA  distributions.

Finally, the results are shown to hold also for distributions with increasing
failure rates  (IFR), and extensions to the  k - sample problem are given.

2.  <u>THE TWO-SAMPLE LIFE-TESTING PROBLEM.</u>  $X_1 ,\ldots, X_m$ and $Y_1 ,\ldots, Y_n$
are independent random samples from populations with life distributions  F  and
G .  $N = m + n$ , $F(t) = H(t/\theta_N)$ , $G(t) = H(t/\gamma_N)$ , H  has the density  h  and
is  IFRA , i.e., $H(0) = 0$  and for each  $t > 0$ ,

$$(2.1) \qquad\qquad \frac{d}{dt} \left\{ - \log \left[ 1 - H(t) \right] / t \right\} = \frac{\ln [1 - H(t)]}{t^2} + \frac{h(t)}{t[1 - H(t)]} \geq 0$$

$r_1 ,\ldots, r_m$  denote the ranks of the  x's

in the combined sample. The level $\alpha$ Savage (1956) test $\psi_N$ of $H_o: \Delta_N = (\theta_N/\gamma_N) = 1$ against $\Delta_N > 1$ rejects for large values of the statistic

$$(2.2) \qquad S_N = \frac{1}{m} \sum_{i=1}^{m} - \ln (1 - \frac{r_i}{N+1})$$

It is assumed throughout that

$$(2.3) \qquad 0 < \lim_{N \to \infty} (m/N) = \lambda < 1$$

Let $0 \leq c \leq \infty$ and consider sequences of alternatives $\{\Delta_N\}$ satisfying

$$(2.4) \qquad \lim_{N \to \infty} N^{\frac{1}{2}} (\Delta_N - 1) = c$$

Then the asymptotic power function $\beta (c;\varphi,H)$ of a test $\varphi_N$ is defined as the limit of the power for such alternatives, i.e.

$$(2.5) \qquad \beta (c;\varphi,H) = \lim_{N \to \infty} \inf \beta_N (\varphi_N| H)$$

where $\beta_N( \varphi_N|H) = E(\varphi_N|F_N,G_N)$ denotes the power of $\varphi_N$ when $F_N(t) = H(t/\theta_N)$, $G_N(t) = H(t/\gamma_N)$ and $\Delta_N = \theta_N/\gamma_N$ satisfies (2.4).

Let $\Phi$ be the standard normal distribution function. Then the results of Chernoff and Savage (1958), Fatou's Lemma, and a few computations yield

Lemma 2.1. Suppose $H$ has a density $h$ and that $H(o) = 0$, then the asymptotic power function of the level $\alpha$ Savage test $\psi_N$ is given by

$$(2.6) \qquad \beta(c;\psi,H) = \Phi \left( \Phi^{-1}(\alpha) + c [\lambda(1-\lambda)]^{\frac{1}{2}} \int_0^{\infty} \frac{th(t)}{1-H(t)} d H(t) \right)$$

The next result shows that $\psi$ and the exponential distribution $K_\sigma(x) = 1 - \exp (- x/\sigma)$ is a saddle point for the asymptotic power function $\beta(c;\varphi,H)$. In other words, $\psi$ is worst for the exponential distribtuion, but is better than all other tests for this distribution.

Theorem 2.1. For all $o \leq c \leq \infty$ and all $\sigma > o$,

$$(2.7) \qquad \sup_\varphi \beta(c;\varphi,K_\sigma) = \beta(c;\psi,K_\sigma) = \inf_H \beta(c;\psi,H)$$

where H ranges over the class of IFRA distribtuions with a density, and $\varphi_N$ ranges over the class of all level $\alpha$ tests.

Proof. The left hand equality was proved by Capon (1961) by essentially comparing $\psi_N$ with the Neyman - Pearson test for $K_\sigma$. To prove the right equality, note that (2.1) yields

$$(2.8) \qquad \frac{th(t)}{1-H(t)} \geq - \ln [1-H(t)] \quad,$$

thus

$$\int_o^\infty \frac{th(t)}{1-H(t)} \, d\, H(t) \geq \int_o^\infty - \ln [1-H(t)] \, d\, H(t) = 1$$

The equality signs hold if and only if H has a constant failure rate average, i.e., if and only if H is exponential, thus

Corollary 2.1. If H is IFRA , has a density, and is not exponential, then

$$(2.9) \qquad \beta(c;\psi,K_\sigma) < \beta(c;\psi,H)$$

The minimax property of the Savage statistic now follows at once from Theorem 2.1.

Theorem 2.2. The level $\alpha$ Savage test $\psi_N$ is asymptotically minimax over the class $\Omega$ of all IFRA distributions with a density, i.e. if H ranges over $\Omega$, then

$$(2.10) \qquad \inf_H \beta(c;\psi,H) \geq \inf_H \beta(c;\varphi,H)$$

for all level $\alpha$ tests $\varphi_N$ .

## Remarks

(i) H is said to have increasing failure rate (IFR) [1] if $H(0)=0$ and $h(t)/[1-H(t)]$ is nondecreasing in $t > 0$. The class of IFR distributions contains the class of exponential distributions and is contained in the class of IFRA distributions. It follows that Theorem 2·1, Corollary 2.1 and Theorem 2.2 holds also for this class.

(ii) The results of this section are stronger than the minimax results of [6] in the sense that no conditions such as bounds on the Kolmogorov distances or variances of the distributions are needed.

(iii) The lim inf in (2.5) can be replaced by a limit if one assumes conditions as in Lemma 3 of Hodges and Lehmann (1963). The results hold if lim inf is replaced by lim sup or partially replaced by lim sup as in [6].

(iv) An asymptotically equivalent form of the Savage statistic is (see [9,p. 1127]),

$$\sum_{i=1}^{m} J_o(r_i) \text{ , where}$$

(2.11)
$$J_o(k) = \sum_{j=N-k+1}^{N} \frac{1}{j}$$

(v) The results in this section hold if one, instead of considering level $\alpha$ tests, considers $\varphi_N$ with asymptotic level $\alpha$, i.e. tests for which $E(\varphi_N | \theta = \gamma) \to \alpha$ as $N \to \infty$.

(vi) The one-sided alternative $\Delta > 1$ can be replaced by the two-sided alternative $\Delta \neq 1$.

(vii) For the k-sample problem with model $F_i(x) = H(x/[1+\theta c_i])$; $i=1,\ldots,$ k; the Puri (1964) extention of the Savage statistic is asymptotically minimax for testing $H_o^k : \theta = 0$ against $\theta > 0$ (or $\theta \neq 0$).

## 3. EFFICIENCY OF THE BEST TEST FOR EXPONENTIAL MODELS.

When H equals an exponential distribution $K_\sigma(t) = 1 - \exp(-t/\sigma)$, then the uniformly most powerful level $\alpha$ test [7] $\varphi_N^*$ of $\theta = \gamma$ against $\theta > \gamma$ rejects when

$$(3.1) \qquad T = \frac{1}{m} \sum_{i=1}^{m} X_i \Big/ \frac{1}{n} \sum_{i=1}^{n} Y_i > F_{2m,2n}(\alpha)$$

where $F_{2m,2n}(\alpha)$ is obtained from the tables of the F distribution. In this section the performance of T is investigated when the assumption of exponentiality is violated and H is an IFRA distribution.

Upon writing

$$(3.2) \qquad \sqrt{N}\,(T - \Delta) = \sqrt{N}\,(\overline{X} - \Delta\overline{Y})\,/\,\overline{Y} \ ,$$

it is clear that $\sqrt{N}\,(T - \Delta)$ has an asymptotic normal distribution with mean zero and variance

$$\sigma^2(T) = \Delta^2\,\sigma^2(H)\,/\,\lambda(1-\lambda)\,\mu^2(H) \quad \text{where}$$

(3.3)

$$\mu(H) = \int_0^\infty t\,dH(t) \quad \text{and} \quad \sigma^2(H) = \int_0^\infty t^2\,dH(t) - \mu^2(H)\ .$$

When H is exponential, then $\sigma^2(H) = \mu^2(H)$. It follows that when H is such that $\sigma^2(H) \neq \mu^2(H)$, then $\varphi_N^*$ does not have level $\alpha$ asymptotically, in fact

$$(3.4) \qquad E(\varphi_N^* \mid \theta = \gamma) \to \Phi\,(\Phi^{-1}(\alpha)\,\mu(H)/\sigma(H)) \quad \text{as} \quad N \to \infty$$

Thus when $\alpha < \frac{1}{2}$ and $\mu(H) > \sigma(H)$, then the asymptotic level of $\varphi_N^*$ is less

.

than $\alpha$. Barlow, Marshall and Proschan (1963) have essentially shown that for IFRA distributions, $\mu(H) \geq \sigma(H)$. The asymptotic power function of $\varphi_N^*$ is

$$(3.5) \qquad \beta(c;\varphi^*, H) = \Phi(\{\Phi^{-1}(\alpha) + c[\lambda(1-\lambda)]^{\frac{1}{2}}\} \mu(H)/\sigma(H))$$

$\varphi_N^*$ can easily be modified to have asymptotic level $\alpha$ by dividing

$\sqrt{N}$ $(T-1)$ by a consistent estimate of

$r(H) = \sigma(H)/\mu(H);$ e.g.

$\hat{r}(H) = \hat{\sigma}(H)/\hat{\mu}(H)$ with

$\hat{\mu}(H) = \frac{1}{N} (\Sigma x_i + \Sigma y_i)$ and

$\hat{\sigma}(H) = \frac{1}{N} (\Sigma x_i^2 + \Sigma y_i^2) - \hat{\mu}^2(H)$ .

For this test, $\hat{\varphi}_N$ , one has

$$(3.6) \qquad \beta(c;\hat{\varphi},H) = \Phi(\Phi^{-1}(\alpha) + c[\lambda(1-\lambda]^{\frac{1}{2}}\mu(H)/\sigma(H))$$

Since $\mu(H) \geq \sigma(H)$ [1] when $H$ is IFRA, $\mu(k_\sigma) = \sigma(k_\sigma)$ , and since $\beta(c;\psi,k_\sigma) = \beta(c,\hat{\varphi},k_\sigma)$, then (2.7) and remark (i) of Section 2 yields

<u>Theorem 3.1.</u> <u>For all</u> $o \leq c \leq \infty$ <u>and all</u> $\sigma > o$ ,

$$(3.7) \qquad \sup_\varphi \beta(c;\varphi,k_\sigma) = \beta(c;\hat{\varphi},k_\sigma) = \inf_H \beta(c,\hat{\varphi},H)$$

<u>where</u> $H$ <u>ranges over the class of</u> IFRA <u>distributions and</u> $\varphi_N$ <u>ranges over the class of all tests with asymptotic level</u> $\alpha$ .

Thus $\hat{\varphi}_N$ is asymptotically minimax in the sense of Theorem 2.2 for the class of IFRA distributions and the class of tests with asypmtotic level $\alpha$ . To see that this is not true for $\varphi_N^*$ , let $H$ be an IFRA distribution with $\mu(H) > \sigma(H)$ , then for each $\alpha < \frac{1}{2}$ ,

$$(3.8) \qquad \beta(c;\varphi^*,H) < \beta(c;\hat{\varphi},k_\sigma) \quad \text{for} \quad 0 \leq c < \sigma(H)/\mu(H) .$$

Let Pitman asymptotic efficiency be as defined in [10]. It follows from (2.6) and (3.6) that the Pitman efficiency of the Savage test $\psi_N$ to the modified classical test $\hat{\varphi}_N$ is

$$(3.9) \qquad e(\psi, \hat{\varphi}) = \frac{\sigma^2(H) \left[ \int_0^\infty xq(x)dH(x) \right]^2}{\mu^2(H)}$$

where $q(x) = h(x)/[1-H(x)]$ is the failure rate of $H$.

The Weibull distribution is defined by

$$(3.10) \qquad \hat{H}(x) = 1 - e^{-ax^b} \quad ; \ a,b > 0; \ x \geq 0$$

If $\mu_k$ denotes the $k$th moment about zero, then

$$(3.11) \qquad \mu_k = a^{-\frac{k}{b}} \Gamma\left(\frac{k}{b} + 1\right), \ q(x) = ab \, x^{b-1},$$

$$\text{and} \quad \int_0^\infty xq(x)d\hat{H}(x) = ab \, \mu_b = b$$

Thus for the Weibull distribution

$$(3.12) \qquad e_b(\psi, \hat{\varphi}) = \frac{b^2 \left[ \Gamma\left(\frac{2}{b} + 1\right) - \Gamma^2\left(\frac{1}{b} + 1\right) \right]}{\Gamma^2\left(\frac{1}{b} + 1\right)}$$

For $b = 1$, the Weibull distribution coinsides with exponential distribution and $e_1(\psi, \hat{\varphi}) = 1$. For $b = 2$, one has the linear failure rate $q(x) = 2ax$ and (3.12) becomes

$$(3.13) \qquad e_2(\psi, \hat{\varphi}) = \frac{16}{\pi} - 4 \doteq 1.093$$

Using L' Hospitals rule, one finds that

$$(3.14) \qquad \lim_{b \to \infty} e_b(\psi, \hat{\varphi}) = \infty$$

When $b < 1$, the failure rate is decreasing. Stirling's approximation shows that if $k = (1/b)$ is large, then (3.12) is approximately $2^{2k+\frac{1}{2}} k^{k-2} e^{-k} - k^{-2}$, and that

$$(3.15) \qquad \lim_{b \to 0} e_b(\psi, \hat{\varphi}) = \infty$$

If $k$ is an integer, then

$$(3.16) \qquad e_b(\psi, \hat{\varphi}) = \frac{(2k)!}{k^2 (k!)^2} - \frac{1}{k^2}$$

For $k = 2$ and $3$ (3.16) becomes $1.25$ and $(19/9) = 2.11$ respectively.

It is easy to show ( [15] and [9] ) that $\psi_N$ is asymptotically most powerful and locally most powerful for the Weibull distribution. Thus

$$(3.17) \qquad e_b(\psi, \varphi) \geq 1 \quad \text{for all} \quad b > 0$$

and for all test $\varphi_N$ for which this efficiency is computable. In particular (3.17) holds for $\hat{\varphi}_N$. Thus the Savage test $\psi_N$ is uniformly more efficient than the adjusted classical test $\hat{\varphi}_N$ for the Weibull distribution. Moreover, the Savage test is much better when the failure rate parameter $b$ is large or close to zero. It is conjectured that the Savage statistic is uniformly more efficient than $\hat{\varphi}_N$ for all distributions with monotone failure rates.

4. CENSORED SAMPLES. Fix $M \leq N$ and wait until a total of $M$ $X$'s and $Y$'s have failed. Let $m' \leq m$ be the number of $\underline{X}$'s observed, then the ranks $r_1, \ldots, r_{m'}$ can be computed from the data. The Gastwirth (1965) modified Savage statistic is

$$(4.1) \qquad S_M = -\frac{1}{M}\left[\ \sum_{i=1}^{m'} \ln\left(1 - \frac{r_i}{N+1}\right) + m' + (m-m')\ln\left(1 - \frac{M}{N+1}\right)\right]$$

It is assumed that

$$(4.2) \qquad\qquad o < \lim_{N \to \infty} (M/N) = p < 1$$

The asymptotic power function of the level $\alpha$ test $\psi_M$ that rejects for large values of $S_M$ can be computed using [9] and [8]. One gets

$$(4.3) \qquad \beta(c;\psi_p,H) = \Phi\left(\Phi^{-1}(\alpha) + c[\lambda(1-\lambda)/p]^{\frac{1}{2}} \int_0^{H^{-1}(p)} \frac{th(t)}{1-H(t)}\, dH(t)\right)$$

From (2.8), it follows that when $H$ ranges over the class of IFRA distributions, then

$$\inf_H \beta(c;\psi_p,H) = \beta(c;\psi_p,K_\sigma)$$

$$(4.4)$$

$$= \Phi\left(\Phi^{-1}(\alpha) + c[\lambda(1-\lambda)/p]^{\frac{1}{2}} [p+\ln(1-p)(1-p)]\right)$$

Since Hájek (1962) and Gastwirth (1965) has shown that

$$(4.5) \qquad\qquad \beta(c;\psi_p,K_\sigma) \geq \beta(c;\varphi,K_\sigma)$$

for all level $\alpha$ tests $\varphi_N$, then the results of Section 2 holds for $\psi_M$.

5. ASYMPTOTIC UNIQUENESS. Stein (1956) and Hájek (1962) has shown that one can obtain asymptotically optimal statistics by estimating the underlying distribution. Although these statistics are impractical, they show that one can not hope for asymptotic uniqueness in the class of all tests with asymptotic level $\alpha$.

Consider the class of one-sided level $\alpha$ rank tests $\mathcal{J}$ [5] based on statistics of the form

(5.1) $$T_M = T_M(J_N) = \frac{1}{m} \sum_{i=1}^{m} J_N\left(\frac{r_i}{N+1}\right)$$

where there exist a function $J$ which is continuous except for possibly a finite number of jump discontinuities and which satisfies

(5.2) $$\int_0^1 J^2(u)\,du < \infty \quad \text{and} \quad \lim_{N\to\infty} \int_0^1 [J_N(u)-J(u)]^2\,du = 0$$

and the conditions of Comment 3.8 of Hájek (1962). Let $\mathcal{F}'$ be the class of IFRA distributions $H$ with a density $h$ which has the Radon - Nykodim derivative $h'$ with respect to Lebesque measure and satisfies

(5.3) $$\int_0^\infty [x^2 h'(x)/h(x)]^2\,dH(x) < \infty \quad.$$

For these classes one has

Theorem 5.2. The Savage - Gastwirth test $\psi_p$ is asymptotically uniquely minimax for $\mathcal{J}$ and $\mathcal{F}'$, i.e., if $\varphi_0 = \varphi_0(J_N) \epsilon \mathcal{J}$, if $H$ ranges over $\mathcal{F}'$, and if

(5.4) $$\inf_H \beta(c;\varphi_0,H) \geq \inf_H \beta(c;\varphi,H)$$

for all $\varphi \epsilon \mathcal{J}$, then there exists constants $a_N$ and $b_N$ such that

(5.5) $$\sqrt{N}\,[S_M - (a_N T_M(J_N) + b_N)] \to 0$$

in probability as $N \to \infty$ provided (2.3),(4.2) and (2.4) hold with $c < \infty$ .

Proof. (2.7) and (5.4) show that $\beta(c;\varphi_0,K_\sigma) = \beta(c;\psi_p,K_\sigma)$. Thus $\varphi_0$ is asymptotically optimal for $K_\sigma$. From Hájek (1962), it follows that the correlation coefficient satisfies

(5.6) $$\rho_N(S_M,T_M \mid K_\sigma; \Delta = 1) \to 1 \quad \text{as} \quad N \to \infty \quad .$$

This implies that for regression coefficients $a_N$ and $b_N$,

$$(5.7) \qquad E(N[S_M - (a_N T_M + b_M)]^2 \mid K_\sigma; \Lambda = 1) \to 0 \qquad .$$

Since $S_M$ and $T_M$ are distribution free, (5.7) holds not only for $k_\sigma$, but for general $H$. The result now follows from the contiquity arguments of LeCam and Hájek (e.g., [9]).

    **6. ESTIMATION.** Barlow and Prochan (1966) have shown that the estimates of the mean that are optimal for exponential models are not robust for IFR distributions. Here an asymptotically robust estimate of the ratio $\mu_1/\mu_2$ of the means of $X$ and $Y$ is constructed using the methods of Hodges and Lehmann (1963). Write $x = (x_1, \ldots, x_m), y = (y_1, \ldots, y_n)$, $ax = (ax_1, \ldots, ax_n)$ etc., and let

$$(6.1) \qquad s(x,y) = S_M$$

be the Savage - Gastwirth statistic (4.1). $\mu_1/\mu_2 = \theta\mu(H)/\gamma\mu(H) = \theta/\gamma = \Delta$, so one estimates $\Delta$.

    Note that $\sqrt{N} s(X, \Delta Y)$ asymptotically tends to be normally distributed about the point $0$ [9]. Let

$$\Delta^* = \sup \{\Delta: s(x, \Delta y) \geq 0\} \quad \text{and}$$
$$(6.2)$$
$$\Delta^{**} = \inf \{\Delta: s(x, \Delta y) \leq 0\}$$

and define the estimate $\hat{\Delta}$ of $\Delta$ by

$$(6.3) \qquad \hat{\Delta} = \hat{\Delta}(x,y) = \tfrac{1}{2}(\Delta^* + \Delta^{**}) \qquad .$$

Since $s(ax,ay) = s(x,y)$ by the invariance properties of ranks, then

$$(6.4) \qquad \hat{\Delta}(ax,ay) = \hat{\Delta}(x,y) \text{ for all } a > 0, \text{ i.e., } \hat{\Delta} \text{ is scale invariant.}$$

Moreover, using this, the definition (6.2), and noting that $s(x, \Delta y)$ is decreasing in $\Delta$ , one gets

$(6.5)$ $\hat{\Delta}(ax, by) = (a/b)\,\hat{\Delta}(x, y)$ ,

$(6.6)$ $P_{\Delta}(\hat{\Delta}/\Delta \leq t) = P_1(\hat{\Delta} \leq t)$ ,

$(6.7)$ $\Delta^* \leq \Delta^{**}$ ,

$(6.8)$ $P(\Delta^* < t) = P(s(x, ty) < 0)$ ,

$(6.9)$ $P(\Delta^{**} \leq t) = P(s(x, ty) \leq 0)$ ,

$(6.10)$ $P(s(x, ty) < 0) \leq P(\hat{\Delta} \leq t) \leq P(s(x, ty) \leq 0)$ , and

**Lemma 6.1.** If H satisfies (5.3) and H(0) = 0, then

$$\lim_{N \to \infty} P_{\Delta}(\sqrt{N}\,[(\hat{\Delta}/\Delta) - 1] \leq t)$$

$$= \Phi\left( t\,[\lambda(1-\lambda)/p]^{\frac{1}{2}} \int_0^{H^{-1}(p)} \frac{xh(x)}{1-H(x)}\,dH(x) \right) .$$

**Proof.** (6.6) shows that one can let $\Delta = 1$. From (6.10) it follows that

$$\lim_{N \to \infty} P_1(N^{\frac{1}{2}}(\hat{\Delta} - 1) \leq t) = \lim_{N \to \infty} P_1(\hat{\Delta} \leq 1 + tN^{-\frac{1}{2}})$$

$$= \lim_{N \to \infty} P_1(s(\underline{X}, (1 + tN^{-\frac{1}{2}})Y) \leq 0)$$

$$= \lim_{N \to \infty} P_{\Delta_N}(s(\underline{X}, Y) \leq 0)$$

where $\Delta_N = 1/(1 + tN^{-\frac{1}{2}})$ . Since $N^{\frac{1}{2}}(\Delta_N - 1) \to t$ as $N \to \infty$ , the result follows from (4.3).

Lemma 6.1 shows that the asymptotic variance of $\sqrt{N}\,[(\hat{\Delta}/\Delta) - 1]$ is

$(6.11)$ $\qquad V(\hat{\Delta}, H) = 1/[\lambda(1-\lambda)/p]^{\frac{1}{2}} \int_0^{H^{-1}(p)} \frac{th(t)}{1 - H(t)}\,dH(t)$ .

Moreover, (4.4) shows that the maximum asymptotic variance over IFRA distributions is

(6.12)  $\sup_H V(\hat{\Delta},H) = V(\hat{\Delta},K_\sigma) = 1/[\lambda(1-\lambda)/p]^{\frac{1}{2}}[p + \ln(1-p)(1-p)]$

Let $\mathcal{F}'$ be as in Section 5, then the results of the previous sections yield

Theorem 6.1. $\hat{\Delta}$ is asymptotically minimax over $\mathcal{F}'$ and the class $\mathcal{E}$ of scale invariant estimates that are asymptotically normal; i.e., if $V(\tilde{\Delta},H)$ denotes the asymptotic variance of the estimate $\tilde{\Delta} \in \mathcal{E}$, then

(6.13)  $\sup_H \{V(\hat{\Delta},H): H \in \mathcal{F}'\} \leq \sup_H \{V(\tilde{\Delta},H): \mathcal{F}' \in H\}$

$V(\hat{\Delta},H)$ also satisfies the saddle-point inequality

(6.14)  $\sup_H V(\hat{\Delta},H) = V(\hat{\Delta},K_\sigma) = \inf_{\tilde{\Delta}} V(\tilde{\Delta},K_\sigma)$

where $H$ ranges over $\mathcal{F}'$ and $\tilde{\Delta}$ over $\mathcal{E}$.

A different approach to the problem of obtaining asymptotic minimax estimates is given by Huber (1963).

# REFERENCES

[1]  Barlow, R. E., A. W. Marshall, and F. Proschan, "Properties of Probability
     Distributions with Monotone Hazard Rate," The Annals of Mathematical
     Statistics, No. 34, pp 375-389, (1963).

[2]  Barlow, R. E., and F. Proschan, "Exponential Life Test Procedures When
     the Distribution Has Monotone Failure Rate," Submitted, (1966).

[3]  Birnbaum, Z. W., J. D. Esary, and A. W. Marshall, "A Stochastic Characterization
     of Wear-Out for Components and Systems," The Annals of Mathematical
     Statistics, No. 37, pp 816-826, (1966).

[4]  Capon, J., "Asymptotic Efficiency of Certain Locally Most Powerful Rank
     Tests," The Annals of Mathematical Statistics, No. 32. pp 88-100, (1961).

[5]  Chernoff, H., and I. R. Savage, "Asymptotic Normality and Efficiency of
     Certain Nonparametric Tests," The Annals of Mathematical Statistics,
     No. 29, pp 972-994, (1958).

[6]  Doksum, K. A., "Asymptotically Minimax Distribution-Free Procedures,"
     The Annals of Mathematical Statistics, No. 37, pp 619-628, (1966).

[7]  Eilboff, J., and J. Nadler, "On Precedence Life Testing," Technometrics,
     No. 7, pp 359-377, (1965).

[8]  Gastwirth, J. L., "Asymptotically Most Powerful Rank Tests for the Two-
     Sample Problem with Censored Data," The Annals of Mathematical Statistics,
     No. 36, pp 1243-1247, (1965).

[9]  Hájek, J., "Asymptotically Most Powerful Rank-Order Tests," The Annals
     Mathematical Statistics, No. 33, pp 1124-1147, (1962).

[10] Hodges, J. L. Jr., and E. L. Lehmann, "Comparison of the Normal Scores
     and Wilcoxon Tests," Proceedings of the Fourth Berkeley Symposium of
     Mathematical Statistics and Probability, No. 1, Univ. of California
     Press, pp 307-317, (1961).

[11] Hodges, J. L., and E. L. Lehmann, "Estimation of Location Based on Rank
     Tests," The Annals of Mathematical Statistics, No. 34, pp 598-611, (1963).

[12] Huber, P. J., "Robust Estimation of a Location Parameter," The Annals
     Mathematical Statistics, No. 35, pp 73-101, (1963).

[13] Lehmann, E. L., Testing Statistical Hypotheses, John Wiley & Sons, New York,
     (1959).

[14] Puri, M. L., "Asymptotic Efficiency of a Class of C-Sample Tests," The Annals
     Mathematical Statistics, No. 35, pp 102-121, (1964).

[15] Savage, I. R., "Contributions to the Theory of Rank Order Statistics: Two
     Sample Case," The Annals of Mathematical Statistics, No. 27, pp 590-616,
     (1956).

[16] Stein, C., "Efficient Nonparametric Testing and Estimation," Proceedings
     of the Third Berkeley Symposium of Mathematical Statistics and Probability,
     No. 1, University of California Press, pp 187-195, (1956).

## DOCUMENT CONTROL DATA - R&D

*(Security classification of title, body of abstract and indexing annotation must be entered when the overall report is classified)*

| 1. ORIGINATING ACTIVITY (Corporate author) | 2a. REPORT SECURITY CLASSIFICATION |
|---|---|
| University of California, Berkeley | Unclassified |
| | 2b. GROUP |

**3. REPORT TITLE**

"ASYMPTOTICALLY OPTIMAL STATISTICS IN SOME MODELS WITH INCREASING FAILURE RATE AVERAGES"

**4. DESCRIPTIVE NOTES (Type of report and inclusive dates)**

Research Report

**5. AUTHOR(S) (Last name, first name, initial)**

Doksum, Kjell

| 6. REPORT DATE | 7a. TOTAL NO. OF PAGES | 7b. NO. OF REFS |
|---|---|---|
| November 1966 | 15 | 16 |

| 8a. CONTRACT OR GRANT NO. 3656(18) | 9a. ORIGINATOR'S REPORT NUMBER(S) |
|---|---|
| b. PROJECT NO. 042 238 | ORC 66-35 |
| c. Research Project No. WW 041 | 9b. OTHER REPORT NO(S) (Any other numbers that may be assigned this report) |

**10. AVAILABILITY/LIMITATION NOTICES**

Distribution of this document is unlimited.

| 11. SUPPLEMENTARY NOTES | 12. SPONSORING MILITARY ACTIVITY |
|---|---|
| | Mathematical Science Division |

**13. ABSTRACT**

Let $F$ and $G$ be defined by $F(t) = H(\gamma t)$ and $G(t) = H(\theta t)$ where $H$ is unknown and $H(0) = 0$. For testing the equality of the means of $F$ and $G$ in the two-sample problem; it is shown that the Savage (The Annals of Mathematical Statistics (1956) pp 590-615) statistic maximizes the minimum power over IFRA (or IFR) distributions asymptotically. Asymptotic uniqueness holds only in a class of rank tests. The results are extended to censored samples, the problem of estimating the ratio of the means, and the k-sample problem.

| 14. KEY WORDS | LINK A | | LINK B | | LINK C | |
|---|---|---|---|---|---|---|
| | ROLE | WT | ROLE | WT | ROLE | WT |
| Testing<br>Estimation<br>IFRA | | | | | | |

## INSTRUCTIONS

1. **ORIGINATING ACTIVITY:** Enter the name and address of the contractor, subcontractor, grantee, Department of Defense activity or other organization *(corporate author)* issuing the report.

2a. **REPORT SECURITY CLASSIFICATION:** Enter the overall security classification of the report. Indicate whether "Restricted Data" is included. Marking is to be in accordance with appropriate security regulations.

2b. **GROUP:** Automatic downgrading is specified in DoD Directive 5200.10 and Armed Forces Industrial Manual. Enter the group number. Also, when applicable, show that optional markings have been used for Group 3 and Group 4 as authorized.

3. **REPORT TITLE:** Enter the complete report title in all capital letters. Titles in all cases should be unclassified. If a meaningful title cannot be selected without classification, show title classification in all capitals in parenthesis immediately following the title.

4. **DESCRIPTIVE NOTES:** If appropriate, enter the type of report, e.g., interim, progress, summary, annual, or final. Give the inclusive dates when a specific reporting period is covered.

5. **AUTHOR(S):** Enter the name(s) of author(s) as shown on or in the report. Enter last name, first name, middle initial. If military, show rank and branch of service. The name of the principal author is an absolute minimum requirement.

6. **REPORT DATE:** Enter the date of the report as day, month, year; or month, year. If more than one date appears on the report, use date of publication.

7a. **TOTAL NUMBER OF PAGES:** The total page count should follow normal pagination procedures, i.e., enter the number of pages containing information.

7b. **NUMBER OF REFERENCES:** Enter the total number of references cited in the report.

8a. **CONTRACT OR GRANT NUMBER:** If appropriate, enter the applicable number of the contract or grant under which the report was written.

8b, 8c, & 8d. **PROJECT NUMBER:** Enter the appropriate military department identification, such as project number, subproject number, system numbers, task number, etc.

9a. **ORIGINATOR'S REPORT NUMBER(S):** Enter the official report number by which the document will be identified and controlled by the originating activity. This number must be unique to this report.

9b. **OTHER REPORT NUMBER(S):** If the report has been assigned any other report numbers *(either by the originator or by the sponsor)*, also enter this number(s).

10. **AVAILABILITY/LIMITATION NOTICES:** Enter any limitations on further dissemination of the report, other than those imposed by security classification, using standard statements such as:

   (1) "Qualified requesters may obtain copies of this report from DDC."

   (2) "Foreign announcement and dissemination of this report by DDC is not authorized."

   (3) "U. S. Government agencies may obtain copies of this report directly from DDC. Other qualified DDC users shall request through
_____."

   (4) "U. S. military agencies may obtain copies of this report directly from DDC. Other qualified users shall request through
_____."

   (5) "All distribution of this report is controlled. Qualified DDC users shall request through
_____."

If the report has been furnished to the Office of Technical Services, Department of Commerce, for sale to the public, indicate this fact and enter the price, if known.

11. **SUPPLEMENTARY NOTES:** Use for additional explanatory notes.

12. **SPONSORING MILITARY ACTIVITY:** Enter the name of the departmental project office or laboratory sponsoring *(paying for)* the research and development. Include address.

13. **ABSTRACT:** Enter an abstract giving a brief and factual summary of the document indicative of the report, even though it may also appear elsewhere in the body of the technical report. If additional space is required, a continuation sheet shall be attached.

It is highly desirable that the abstract of classified reports be unclassified. Each paragraph of the abstract shall end with an indication of the military security classification of the information in the paragraph, represented as (TS). (S). (C), or (U).

There is no limitation on the length of the abstract. However, the suggested length is from 150 to 225 words.

14. **KEY WORDS:** Key words are technically meaningful terms or short phrases that characterize a report and may be used as index entries for cataloging the report. Key words must be selected so that no security classification is required. Identifiers, such as equipment model designation, trade name, military project code name, geographic location, may be used as key words but will be followed by an indication of technical context. The assignment of links, roles, and weights is optional.

DD FORM 1473 (BACK)
1 JAN 64